CONSERVATION LAWS FOR EULER-LAGRANGE EXTERIOR DIFFERENTIAL SYSTEMS

BEN MCMILLAN

Abstract. The calculus of variations is a powerful tool PDE, allowing detailed analysis when applicable. In this series of 2 talks I will explain how to describe them in a coordinate free manner using exterior differential systems. Using this we will be describe a particularly nice formulation of Noether's theorem describing the space of conservation laws. From this we will be able to derive some nice constraints on minimal surfaces.

Note: A very large portion of this talk is lifted from the first chapter of *Exterior Differential Systems and Euler-Lagrange Partial Differential Equations* by Bryant, Griffiths and Grossman.

1. The Classical variation problem

The first goal of these notes is to explain how the classical variational problems arising in PDE and physics can be rewritten as an exterior differential system. Once this is done we will be in a position to describe Noether's theorem relating symmetries of the problem to conservation laws. We quickly review what is meant by variational problem.

Given a smooth function $L(x, z, p)$ in the variables $x, p \in \mathbb{R}^n$ and $z \in \mathbb{R}$ we can define a functional on the space of functions $C^\infty(U, \mathbb{R})$ for some open, bounded set $U \subset \mathbb{R}^n$ by

$$\mathcal{F}(z) = \int_U L(x, z, \nabla z) dx.$$

Given such a functional, a critical point is one so that $\frac{d}{dt}\mathcal{F}(z + t\epsilon) = 0$ for any $\epsilon \colon U \to \mathbb{R}$ where $\epsilon|_{\partial U} = 0$. The critical points are exactly the functions which satisfy the *Euler-Lagrange equation*

$$\frac{\partial L}{\partial z} - \sum \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial p_i}\right) = 0.$$

For example, if we consider the class of hypersurfaces in $\mathbb{R}^{n+1}$ we can define the area functional which maps a surface to its area. More concretely, we may locally consider graphs of $z \colon U \to \mathbb{R}$ whose restrictions to the boundary agree with a given $z_0 \colon \partial U \to \mathbb{R}$. Setting $dx = dx^1 \wedge \ldots \wedge dx^n$, the area of such a graph is then given by the functional

$$\mathcal{A}(z) = \int_U \sqrt{1 + \|\nabla z\|^2} dx.$$

A function $f(x)$ which respects the boundary condition and whose graph locally minimizes this functional satisfies the Euler-Lagrange equation

$$\sum \frac{\partial}{\partial x_i}\left(\frac{f_{x_i}}{\sqrt{1 + \sum f_{x_i}^2}}\right) = 0$$

It will be profitable to cast this setup in the language of Exterior Differential Systems for several reasons, the most basic one being that we are freed from the need to choose coordinates. For example, in the minimal surface case we are free to consider surfaces which are not globally represented by a graph as above.

2. The Euler-Lagrange System

In the above setup the function L is a function of x and z as well as the first derivatives of z. For this reason, the natural space for our calculations to live on will be $M = J^1(\mathbb{R}^n, \mathbb{R})$, the set of 1-jets of functions on $\mathbb{R}^n$. This space has natural coordinates which we call x^i, z and p_i for $1 \leq i \leq n$ and a natural contact form

$$\theta = dz - p_i dx^i.$$

The *1-jet graph* of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is the submanifold

$$N = \{(x, f(x), \nabla f(x)) \in J^1(\mathbb{R}^n, \mathbb{R})\},$$

It is a standard exercise to see that any such 1-jet graph is an integral manifold of the differential ideal $\mathcal{I} = \{\theta, d\theta\}$. Conversely, an n-dimensional integral manifold N for which the restriction of $dx^1 \wedge \ldots \wedge dx^n$ is never zero will be the 1-jet graph of a function. Indeed, the condition $dx^1 \wedge \ldots \wedge dx^n$ means that we can locally describe N as

$$N = \{(x^i, f(x), p_i(x))\}$$

for some functions f and p_i of x. Then

$$0 = \theta|_N = (dz - p_i dx^i)|_N = df - p_i(x)dx^i = \left(\frac{\partial f}{\partial x^i} - p_i(x)\right) dx^i$$

so that N is the 1-jet graph of f. An integral n-manifold for $\mathcal{I}$ is also known as Legendre for the contact system. Intuitively, the condition that $\theta|_N = 0$ forces the p_i to be the derivatives of z.

With this setup, we define a *Lagrangian* on M to be any n-form $\Lambda \in \Omega^n(M)$. To this we associate a functional on integral manifolds of $\mathcal{I}$ by

$$\mathcal{F}_\Lambda(N) = \int_N \Lambda.$$

For example, if $\Lambda = L(x, z, p)dx$ we recover the classical case described in the first section. Note that while $\mathcal{F}_\Lambda$ could be considered a functional on all n-manifolds in M, we restrict to integral manifolds. One consequence of this restriction is that the functional $\mathcal{F}_\Lambda$ only depends on Λ up to congruence modulo $\{\theta\}$. Indeed, for $\tilde{\Lambda} = \Lambda + \theta \wedge \beta$ we have for any integral manifold N

$$\mathcal{F}_{\tilde{\Lambda}}(N) = \int_N \Lambda + \int_N \theta \wedge \beta = \mathcal{F}_\Lambda(N).$$

The more important object to study turns out to be the derivative $\Pi = d\Lambda$ of Λ, which we now normalize for later use. By symplectic linear algebra[1] we can write always find forms α and β so that

$$\begin{aligned} d\Lambda &= \theta \wedge \alpha + d\theta \wedge \beta \\ &= \theta \wedge (\alpha + d\beta) + d(\theta \wedge \beta) \end{aligned}$$

Thus if we replace Λ with $\Lambda - \theta \wedge \beta$ we may assume, without affecting our functional, that $d\Lambda = \theta \wedge \Psi$ for the n-form $\Psi = \alpha + d\beta$.

To determine the critical values of our functional, suppose we have an n-dimensional manifold N with boundary and a Legendre immersion $\iota\colon N \hookrightarrow M$. A *variation* of ι is a map

$$F\colon N \times (-\epsilon, \epsilon) \to M$$

[1] $d\Lambda$ is an $n+1$ form, so its image in $T_x^*M/\{\theta\}$ cannot be primitive with respect to the symplectic form $d\theta$.

so that $F|_{N\times\{0\}} = \iota$, $F(N\times\{t\})$ is a Legendre submanifold for all $t\in(-\epsilon,\epsilon)$ and so that $F(q,t)$ is independent of t for any $q\in\partial N$. We will denote by N_t the image $F(N\times\{t\})$ and let $F_t(x) = F(x,t)$.

If the Legendre manifold N_0 minimizes the functional $\mathcal{F}_\Lambda$ then for any variation N_t we will have $\frac{d}{dt}\mathcal{F}_\Lambda(N_t) = 0$. Expanding the left hand side we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_\Lambda(N_t) &= \frac{d}{dt}\int_N F_t^*\Lambda \\ &= \int_N \mathcal{L}_{\frac{\partial}{\partial t}} F_t^*\Lambda \\ &= \int_N \tfrac{\partial}{\partial t} \lrcorner F_t^* d\Lambda + \int_N d\left(\tfrac{\partial}{\partial t}\lrcorner F_t^*\Lambda\right) \\ &= \int_N \tfrac{\partial}{\partial t} \lrcorner F_t^* d\Lambda + \int_{\partial N} \tfrac{\partial}{\partial t}\lrcorner F_t^*\Lambda \\ &= \int_N \tfrac{\partial}{\partial t} \lrcorner F_t^* d\Lambda\end{aligned}$$

the last equality holding because $F_t'(\frac{\partial}{\partial t}) = 0$ on the boundary by assumption. Finally, using our normalization assumption that $d\Lambda = \theta\wedge\Psi$ and the fact that $F_t^*\theta = 0$ we see that

$$\begin{aligned}\int_N \tfrac{\partial}{\partial t}\lrcorner F_t^* d\Lambda &= \int_N \tfrac{\partial}{\partial t}\lrcorner F_t^*(\theta\wedge\Psi) \\ &= \int_N (\tfrac{\partial}{\partial t}\lrcorner F_t^*\theta)\Psi\end{aligned}$$

We let $g = (\frac{\partial}{\partial t}\lrcorner F_t^*\theta)|t=0$ and conclude that

$$\left.\frac{d}{dt}\right|_{t=0}\mathcal{F}_\Lambda(N_t) = \int_N g\cdot\iota^*\Psi = 0$$

for a critical point of $\mathcal{F}_\Lambda$. Since this holds for any variation, and hence any g, we conclude that a Legendre manifold N is a critical point exactly if $\iota^*\Psi = 0$. In other words, *critical points of* $\mathcal{F}_\Lambda$ *are exactly the integral manifolds of the ideal* $\mathcal{E} = \{\theta, d\theta, \Psi\}$.

To demonstrate all of this, let us calculate the Euler-Lagrange equations for a classical Lagrangian $\Lambda = L(x,z,p)dx^1\wedge\ldots\wedge dx^n$. We compute

$$\begin{aligned}d\Lambda &= (L_z dz + L_{p_i}dp_i)\wedge dx^1\wedge\ldots\wedge dx^n \\ &= \theta\wedge L_z dx^1\wedge\ldots\wedge dx^n - d\theta\wedge(-1)^{i-1}L_{p_i}dx^1\wedge\ldots\wedge \hat{dx^i}\wedge\ldots\wedge dx^n \\ &= \theta\wedge\alpha + d\theta\wedge\beta.\end{aligned}$$

Now we use the normalization as above to see that

$$\begin{aligned}\Psi &= \alpha + d\beta \\ &= L_z dx + (-1)^i d(L_{p_i}dx^1\wedge\ldots\wedge\hat{dx^i}\wedge\ldots\wedge dx^n)\end{aligned}$$

Then an integral manifold of $\mathcal{E}$ will of the form $N = \{(x, z(x), z_{x^i}(x))\}$ and

$$\begin{aligned}\Psi|_N &= L_z dx^1\wedge\ldots\wedge dx^n + (-1)^i(L_{p_i x^j}dx^j + L_{p_i z}dz + L_{p_i p_j}dp^j)\wedge dx^1\wedge\ldots\wedge\hat{dx^i}\wedge\ldots\wedge dx^n \\ &= (L_z - L_{p_i x^i} - L_{p_i z}z_{x^i} - L_{p_i p_j}z_{x_i x_j})dx \\ &= \left(\frac{\partial L}{\partial z}(x, z(x), z_{x^i}(x)) - \frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial p_i}(x, z(x), z_{x^i}(x))\right)\right)dx.\end{aligned}$$

The coefficient term of the last line is the standard Euler-Lagrange equation and we see that it is zero if and only if N is a critical point for the functional $\mathcal{F}_\Lambda$.

3. The Space of Conservation Laws

Switching gears temporarily, we define a *conservation law* of an exterior differential system $(M, \mathcal{E})$ to be any form $\varphi \in \Omega^{n-1}(M)$ so that $d\varphi \in \mathcal{E}$. Thus a conservation law satisfies $d\varphi|_N = 0$ for any integral manifold of N. This is a concise way representing 'conserved quantities,' and their existence allows us to place strong conditions on the possible integral manifolds.

For example, consider a 1-dimensional variational problem with Lagrangian $L(x, z, p) = \frac{1}{2}mp^2 - U(z)$. In physics this is the system of a single particle of mass m moving along the real axis with a potential field $U(z)$ which is a function of position. The $\frac{1}{2}mp^2$ term is the kinetic energy of the particle and the $U(z)$ term is its potential energy. Later we will see that the independence of the Lagrangian on x gives us the conservation law $E = \frac{1}{2}mp^2 + U(z)$, which represents the total energy of the system. As a conservation law, the function E on the 'phase space' $M = J^1(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}^3$ has the property that $dE|_N = 0$ for any solution. In particular, we see that E is constant along any solution. Note that $dE \neq 0$ as a 1-form on M, so the level sets $E = \mathit{const}$ are hypersurfaces. Any solution must lie within one of these level sets and we have reduced the phase space by 1-dimension.

We now define the space of conservation laws. A conservation law will not tell us anything if φ is already in $\mathcal{E}$ or if it is exact, so we will want to quotient those out. In other words, a conservation law is an $n-1$-form φ so that $d\varphi \in \mathcal{E}$, modulo exact $n-1$-forms and those already in $\mathcal{E}$. Because of this, we define the space of *conservation laws* to be $\mathcal{C} = H^{n-1}(\bar{\Omega}^*)$ where $\bar{\Omega}^k = \Omega^k(M)/\mathcal{E}^k$ and $\mathcal{E}^k = \mathcal{E} \cap \Omega^k(M)$. One might object that a conservation law $[\varphi]$ for which $d\varphi = 0$ is also trivial, and for this reason we will also want to quotient these out. To do this, note that the short exact sequence of chain complexes

$$0 \to \mathcal{E}^* \to \Omega^*(M) \to \bar{\Omega}^* \to 0$$

gives us the long exact sequence

$$\cdots \to H^{n-1}_{dR}(M) \xrightarrow{\pi} \mathcal{C} \to H^n(\mathcal{E}^*) \to \dots$$

We define the space of *proper conservation laws* to be $\overline{\mathcal{C}} = \mathcal{C}/\pi(H^{n-1}_{dR}(M))$. The long exact sequence also gives us an inclusion of $\overline{\mathcal{C}}$ into $H^n(\mathcal{E}^*)$

For a general EDS the calculation of the space of conservation laws is an interesting problem. In the case that our exterior differential system is associated to a Lagrangian on a contact manifold Noether's theorem gives a nice characterization. To explain this we first need to describe the space of symmetries of Λ. A vector field $v \in \mathfrak{X}(M)$ is an infinitesimal symmetry of the Lagrangian if it preserves both the contact form and Λ. We define the space of such vector fields

$$\mathfrak{g}_\Lambda = \{v \in \mathfrak{X}(M) \colon \mathcal{L}_v\theta \equiv 0 (\mathrm{mod}\ \theta), \mathcal{L}_v\Lambda = 0\}.$$

Then we have the

Theorem (Noether)**.** *For a nondegenerate Euler-Lagrange functional Λ with associated Euler-Lagrange system $(M, \mathcal{E}_\Lambda)$ there is an injective linear map*

$$\eta \colon \mathfrak{g}_\Lambda \to H^n(\mathcal{E}^*_\Lambda)$$

so that $\eta(\mathfrak{g}_\Lambda) \subseteq \overline{\mathcal{C}}$.

For a Lagrangian normalized as above so that $d\Lambda = \theta \wedge \Psi$ we may explicitly define

$$\eta(v) = -v \lrcorner \Lambda.$$

Then we easily compute

$$\begin{aligned} d(-v \lrcorner \Lambda) &= \cancel{-\mathcal{L}_v \Lambda} + v \lrcorner (\theta \wedge \Psi) \\ &= \theta(v)\Psi - \theta \wedge (v \lrcorner \Psi) \in \mathcal{E}_\Lambda \end{aligned}$$

To continue the example of a particle in a conservative filed, notice that the Lagrangian $\Lambda = (\frac{1}{2}mp^2 - U(z))dx$ is invariant by the 'time' variable x, as is θ. To use Noether's formula we need to first ensure that Λ is normalized, so we calculate

$$\begin{aligned} d\Lambda &= (mp\ dp - U'(z)dz) \wedge dx \\ &= -mp\ d\theta - U'\ \theta \wedge dx \\ &= \beta d\theta + \theta \wedge \alpha. \end{aligned}$$

Thus we replace Λ with

$$\begin{aligned} \tilde{\Lambda} = \Lambda - \beta\theta &= (\tfrac{1}{2}mp^2 - U(z))dx + mp(dz - p\ dx) \\ &= (-\tfrac{1}{2}mp^2 - U(z))dx + mp\ dz. \end{aligned}$$

Then indeed $d\tilde{\Lambda} = \theta \wedge (m\ dp - U'\ dx)$ and Noether's theorem gives us

$$E = -\partial_x \lrcorner \tilde{\Lambda} = \tfrac{1}{2}mp^2 + U(z)$$

4. Minimal Hypersurfaces

To study minimal hypersurfaces in Euclidean space we will work on the contact manifold over $\mathbb{E}^{n+1}$, defined to be the unit sphere bundle of the tangent bundle, $M = S(T\mathbb{E}^{n+1}) \cong \mathbb{E}^{n+1} \times S^n$ with projection map $x\colon M \to \mathbb{E}^{n+1}$. On M there is a natural (tautological) contact form defined by

$$\theta_{(x,e_0)}(w) = dx(w) \cdot e_0$$

where $w \in T_{(x,e_0)}M$. Since $\mathbb{E}^{n+1}$ is euclidean, we may identify each point $(x, e_0) \in M$ with the hyperplane $\{e_0\}^\perp \subset T_xM$. In this way we may think of M as the bundle of hyperplanes in $\mathbb{E}^{n+1}$. A Legendre n-manifold for θ is the lift of a hypersurface to the hyperplane bundle, like the 1-jet graph situation above.

Although M is our primary object of interest, computations will be much easier on the larger bundle of frames, where we have complete structure equations. By definition, the oriented orthonormal frame bundle $\mathcal{F}$ of $\mathbb{E}^{n+1}$ is the set of oriented orthonormal bases $(e_0, \dots, e_n)$ of $T_x\mathbb{E}^{n+1}$ for each $x \in \mathbb{E}^{n+1}$. This bundle has projection map x as well as framing maps given by $e_i(f) = \tilde{e}_i$ for the frame $(x, \tilde{e}_0, \dots, \tilde{e}_n)$. We can also consider $\mathcal{F}$ as a bundle over M with projection map $x \times e_0$. On $\mathcal{F}$ we have the canonical 1-forms ω^a and ω^a_b given by

$$\omega^i = dx \cdot e_i$$

$$\omega^i_j = de_j \cdot e_i = -\omega^j_i$$

as well as the structure equations

$$d\omega^i = -\omega^i_j \wedge \omega^j$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j.$$

Notice that ω^0 is the pullback of θ by the projection $\mathcal{F} \to M$. On $\mathcal{F}$ the structure equation $d\omega^0 = -\omega^0_a \wedge \omega^a$ shows that $\omega^0 \wedge (d\omega^0)^n \neq 0$. Since the projection $\mathcal{F} \to M$ is a submersion the same must hold for θ, and the claim that it is a contact form is proven.

With this set up, we may define an n-form on $\mathcal{F}$ by

$$\tilde{\Lambda} = \omega^1 \wedge \ldots \wedge \omega^n.$$

Given a section $\sigma\colon M \to \mathcal{F}$ we may define $\Lambda = \Lambda_\sigma = \sigma^*\tilde{\Lambda}$ and this is well defined independent of our choice σ. Indeed, any other section η differs from σ by a multiplication with a section of $M \times SO(n)$, so that $\eta = m \cdot \sigma$. Then we will have the relation

$$\sigma^*\omega^i = (m^{-1})^i_j \eta^*\omega^j$$

so that

$$\Lambda_\sigma = \det(m^{-1})\Lambda_\eta.$$

The functional

$$\mathcal{F}_\Lambda(N) = \int_N \Lambda$$

on Legendre submanifolds of N is exactly the area functional for hypersurfaces in $\mathbb{E}^{n+1}$. Indeed, given a Legendre manifold N of (M,θ) and a section $\sigma\colon M \to \mathcal{F}$, the forms $\sigma^*\omega^1, \ldots, \sigma^*\omega^n$ give an orthonormal framing when restricted to N. Their product is then equal to the volume form on N as well as to Λ.

It is easy to check from the structure equations that[2]

$$d\Lambda = -\theta \wedge \omega^0_i \wedge \omega_{(i)}$$

so that Λ is already normalized with $\Psi = -\omega^0_i \wedge \omega_{(i)}$. From what was shown earlier, minimal hypersurfaces in $\mathbb{E}^{n+1}$ are in bijection with integral manifolds of the ideal

$$\mathcal{I} = \{\theta, d\theta, \omega^0_i \wedge \omega_{(i)}\}$$

which satisfy the independence condition $\omega^1 \wedge \ldots \wedge \omega^n|_N \neq 0$.[3] If N is an integral manifold then the form $d\theta = -\omega^0_i \wedge \omega^i$ is zero on N and the Cartan lemma tells us that, restricted to N,

$$\omega^0_i = h_{ij}\omega^j, \qquad h_{ij} = h_{ji}.$$

The h_{ij} are the coefficients of the second fundamental form of N. Plugging these equations into Ψ we see that

$$0 = \Psi|_N = (\sum_i h_{ii})\omega^1 \wedge \ldots \wedge \omega^n,$$

so the (local) area minimizing hypersurfaces are exactly those of mean curvature 0.

The group of Euclidean symmetries on $\mathbb{E}^{n+1}$ (translations and rotations) induce a 'diagonal' action on $\mathcal{F}$ and the forms ω^i are invariant under this action. Indeed, if we fix a base point and framing for $\mathbb{E}^{n+1}$ then we can identify $\mathcal{F}$ with the group $ASO(3)$ of Euclidean motions. Under this identification the forms ω^i, ω^i_j are the components of the Maurer-Cartan form of $ASO(3)$. By definition these forms are invariant under left multiplication. Then the pulldowns θ and Λ are also invariant under the diagonal action on M. From the action we get a map

$$\mathfrak{aso}(3) \hookrightarrow \mathfrak{g}_\Lambda \subset \mathfrak{X}(M).$$

[2]The notation $\omega_{(i)}$ denotes the form $(-1)^{i-1}\omega^1 \ldots \wedge \hat{\omega}^i \wedge \ldots \wedge \omega^n$. The forms ω^0_i and ω^i are not well defined on M, but their pullback under any section is, and the product $\omega^0_i \wedge \omega_{(i)}$ does not depend on the choice of section. From here on I will identify the ω^0_i, ω^i with their pullbacks by some section with no further comment.

[3]It is sort of a coincidence that the independence condition is equal to Λ, which is to say that for any other Lagrangian we would still use the independence condition $\omega^1 \wedge \ldots \wedge \omega^n$

By Noether's theorem we may identify elements of $\mathfrak{aso}(3)$ with conservation laws of Λ.

For example, suppose $v \in \mathfrak{aso}(3)$ is an infinitesimal translation. We can write this on the frame bundle as

$$v|_{\mathcal{F}} = Ae_0 + A^i e_i$$

where the A, A^i are the coefficients of v in the basis $e_0, \ldots, e_n$. Then Noether's formula gives us the conservation law

$$\varphi = v_{\mathcal{F}} \lrcorner \Lambda = A^i \omega_{(i)}.$$

For an integral manifold N with $\omega^1 \wedge \ldots \wedge \omega^n \neq 0$, the ω^i give an orthonormal coframing and we may write

$$\varphi|_N = A^i(*\omega^i) = * \langle v, dx \rangle$$

where $*$ is the Hodge star operator for N. In fact, we may still use the Hodge star without reference to a particular integral N because any point of M represents a hyperplane. Purely algebraically, at the point $q = (x, e_0) \in M$ representing $H = \{e_0\}^\perp$ we define $\varphi_q = *(\langle v, dx \rangle |_H)$

As we vary the translation vector v we get a linear map $\varphi \in (\mathbb{E}^{n+1})^* \otimes \overline{\mathcal{C}}$. Identifying $\mathbb{E}^{n+1}$ with its dual we may think of φ as an $\mathbb{E}^{n+1}$ valued conservation law, written $*dx$.

For an integral manifold N with smooth boundary this tells us that

$$0 = \int_N d\varphi = \int_{\partial N} *dx.$$

To interpret this, let us adapt our framing so that

- e_0 continues to be the oriented normal to N.
- e_n is the outward normal to ∂N within N.
- $e_1, \ldots, e_{n-1}$ is tangent to ∂N.

Written this way,

$$dx = e_0\omega^0 + \sum_{i=1}^{n-1} e_i\omega^i + e_n\omega^n,$$

and restricted to N we have

$$*dx = \sum_{i=1}^{n-1} e_i\omega_{(i)} + e_n\omega_{(n)}.$$

Finally, on ∂N the form ω^n vanishes and we are left with

$$*dx|_{\partial N} = \pm e_n\omega^1 \wedge \omega^{n-1}.$$

The vector e_n is outward normal and $\omega^1 \wedge \omega^{n-1}$ is the area form on ∂N, se we conclude that the average of the outward normals on the boundary of a minimal surface must be zero.